Solution 10

Supplementary Problems

1. Consider the parametric surface

$$\mathbf{r}(u,v) = (u+6v, -2u-12v+5, -1), \quad (u,v) \in [0,1]\times[0,1] \ .$$

Is it a smooth surface? Describe its image. Recall that by definition a parametric surface is smooth if $\mathbf{r}$ is continuously differentiable and $\mathbf{r}_u \times \mathbf{r}_v$ is linearly independent in the interior of the region of definition.

Solution. $\mathbf{r}_u \times \mathbf{r}_v = (1,-2,0)\times(6,-12,0) = 0$, hence this parametric surface is not smooth (or regular). In fact, the image of this parametric surface degenerates into the straight line $(0,5,-1)+t(1,-2,0)$, $t\in\mathbb{R}$.

2. Let S be the surface of revolution obtained by rotating $\mathbf{r}(t) = (x(t), z(t)), x(t) > 0, t \in [a,b]$ around the z-axis. Show that its surface area is given by

$$2\pi \int_a^b x(t)\sqrt{x'^2(t)+z'^2(t)}\, dt \ .$$

Derive this formula using Riemann sum approach. Hint: Consider the cross sections along the z-axis.

Solution. The surface area is equal to

$$\int_0^{2\pi}\int_a^b f(z)\sqrt{1+f'^2(z)}\, dt\ d\alpha = 2\pi\int_a^b f(z)\sqrt{1+f'^2(z)}\, dz \ .$$

In fact, cutting up the surface along the z-axis, S can be obtained by summing up the surface area of all cross sections. These cross sections are circles of radius $f(z)$, so their surface area is $2\pi f(z)\times\Delta s$ which is approximately $2\pi f(z)\sqrt{1+f'^2(z)}\Delta z$.

Note. In lecture, this formula was derived by calculating $|\mathbf{r}_t \times \mathbf{r}_\alpha|$ where

$$\mathbf{r}(t,\alpha) = (x(t)\cos\alpha, x(t)\sin\alpha, z(t)) \ .$$